Formal and analytic integrability of the Lorenz system

2005 J. Phys. A: Math. Gen. 38 2681

(http://iopscience.iop.org/0305-4470/38/12/010)

J. Phys. A: Math. Gen. **38** (2005) 2681–2686 doi:10.1088/0305-4470/38/12/010

Formal and analytic integrability of the Lorenz system

Jaume Llibre[1] **and Clàudia Valls**[2]

[1] Departament de Matemàtiques, Universitat Autònoma de Barcelona, 08193 Bellaterra, Barcelona, Spain
[2] Departamento de Matemática, Instituto Superior Técnico, 1049-001 Lisboa, Portugal

E-mail: jllibre@mat.uab.es and cvalls@math.ist.utl.pt

Received 24 September 2004, in final form 18 January 2005
Published 9 March 2005
Online at stacks.iop.org/JPhysA/38/2681

Abstract
The well-known Lorenz system can be written as $\dot{x} = s(y-x)$, $\dot{y} = rx-y-xz$ and $\dot{z} = -bz+xy$. Here, we study the first integrals of the Lorenz system that can be described by formal power series. In particular, if $s \neq 0$ and, either b is not a negative rational number, or b is a negative rational number and $k_1 b + k_2(1+s) \neq 0$, for all k_1 and k_2 non-negative integers with $k_1 + k_2 > 0$, then the Lorenz system has no analytic first integrals in a neighbourhood of the origin.

PACS numbers: 04.25.Nx, 02.30.Hq, 02.40.−k, 02.40.Ma
Mathematics Subject Classification: 34C05, 34A34, 34C14

1. Introduction

The Lorenz system (see [9]):

$$\dot{x} = s(y-x), \qquad \dot{y} = rx - y - xz, \qquad \dot{z} = -bz + xy, \tag{1}$$

is a famous dynamical model (see for instance [10]), where x, y and z are real variables; and s, r and b are real parameters. This system has been intensively investigated as a dynamical system (see for instance [14]), mainly for studying its strange attractors, the more classical one appears for the parameter values $s = 10$, $b = 8/3$ and $r = 28$. From the point of view of integrability it was also intensively studied using different integrability theories (for example, see [1, 3–7, 12, 13, 15–18]). But in this paper we are interested in its formal power series first integrals and in its analytical first integrals.

The associated vector field of the Lorenz system is

$$X = s(y-x)\frac{\partial}{\partial x} + (rx - y - xz)\frac{\partial}{\partial y} - (bz - xy)\frac{\partial}{\partial z}. \tag{2}$$

A *first integral* of system (1) is a non-constant function $H = H(x, y, z)$ satisfying

$$XH = s(y-x)\frac{\partial H}{\partial x} + (rx - y - xz)\frac{\partial H}{\partial y} - (bz - xy)\frac{\partial H}{\partial z} = 0.$$

Let H_1 and H_2 be first integrals of the Lorenz system. They are *independent* if the one-forms $\mathrm{d}H_1$ and $\mathrm{d}H_2$ are linearly independent over a full Lebesgue measure subset of the common definition domain of H_1 and H_2. By definition, we say that system (1) is *integrable* if it admits two-independent first integrals.

The following result due to Poincaré [11] is well-known; for a proof, see for instance [2]. We will use it later on.

Theorem 1. *We denote by A the Jacobian matrix of an analytic vector field $X(x)$ at $x = 0$. If the eigenvalues $\lambda_1, \lambda_2, \ldots, \lambda_n$ of A do not satisfy any resonance conditions of the form*

$$\sum_{i=1}^{n} k_i \lambda_i = 0, \qquad k_i \in \mathbb{Z}^+, \qquad \sum_{i=1}^{n} k_i > 0,$$

then the vector field $X(x)$ does not have analytic first integrals in a neighbourhood of the origin.

For a generalization of theorem 1 to a matrix A with a zero eigenvalue see [8]. In this paper $\mathbb{Z}^+$ denotes the set of non-negative integers.

2. Main results

First we prove the next preliminary result which will be necessary for proving theorem 4.

Proposition 2. *Assume that $s = 0$ and b is not a negative rational. If $f = f(x, y, z)$ is a formal power series first integral of the Lorenz system (1), then f is a formal power series in the variable x.*

Proof. Let $f = f(x, y, z)$ be a formal power series first integral of system (1) with $s = 0$. Then, we can write it as

$$f = \sum_{n \geqslant 0} f_n(y, z) x^n = \sum_{k,l,n \geqslant 0} f_{k,l,n} y^k z^l x^n.$$

Letting $s = 0$ in equation (1), we conclude that f satisfies

$$Xf = (rx - y - xz)\frac{\partial f}{\partial y} - (bz - xy)\frac{\partial f}{\partial z} = 0. \tag{3}$$

We will proceed by induction and will prove that for any integer $N \geqslant 0$, $f_N(y, z)$ is constant and equal to $f_{0,0,N}$. This will imply that

$$f = \sum_{n \geqslant 0} f_n(y, z) x^n = \sum_{n \geqslant 0} f_{0,0,n} x^n = f(x),$$

which obviously will finish the proof of the proposition.

We start by proving that $f_0(y, z) = f_{0,0,0}$. To do it, let $x = 0$ in (3). Then, since

$$f_0(y, z) = \sum_{k,l \geqslant 0} f_{k,l,0} y^k z^l,$$

we have

$$-\sum_{k,l \geqslant 0} k f_{k,l,0} y^k z^l - b \sum_{k,l \geqslant 0} l f_{k,l,0} y^k z^l = 0,$$

which yields

$$\sum_{k,l \geqslant 0} (k + bl) f_{k,l,0} y^k z^l = 0. \tag{4}$$

Since by hypothesis b is not a negative rational, we have that $k + bl \neq 0$ for all $k, l \geqslant 0$ with $k + l \geqslant 1$. Then, from (4) we have $f_{k,l,0} = 0$ for all $k, l \geqslant 0$ and $k + l \geqslant 1$. That is, $f_0(y, z) = f_{0,0,0}$. So, the hypothesis of induction is proved for $N = 0$.

Now, we assume that it is true for $N - 1$ (i.e. $f = \sum_{n=0}^{N-1} f_{0,0,n} x^n + \sum_{n \geqslant N} f_n(y, z) x^n$), and we will prove it for N. Clearly, by the induction hypothesis,

$$f = \sum_{k=0}^{N-1} f_{0,0,k} x^k + x^N \sum_{k,l \geqslant 0, n \geqslant N} f_{k,l,n} y^k z^l x^{n-N}.$$

Then, using this form of f and computing the terms in (3) of degree N in x, we obtain

$$-\sum_{k,l \geqslant 0} k f_{k,l,N} y^k z^l - b \sum_{k,l \geqslant 0} l f_{k,l,N} y^k z^l = 0.$$

Then, using the same arguments as in the case $N = 0$, it follows that $f_{k,l,N} = 0$ for all $k, l \geqslant 0$ and $k + l \geqslant 1$. Then, $f_N(y, z) = f_{0,0,N}$. Thus, by the induction process the proposition is proved. □

The main results of this paper are the following ones.

Proposition 3. *If $s = 0$ then the Lorenz system (1) is integrable with the two first integrals*

$$H_1 = x \quad \textit{and} \quad H_2 = F_1(x, y, z) \exp\left(-2 \arctan \frac{F_2(x, y, z)}{F_3(x)}\right), \tag{5}$$

where

$$\begin{aligned} F_1 = {} & x(r^2x^3 - (1 + b)rx^2y + bxy^2 + x^3y^2 + b(b - 1)rxz \\ & - 2rx^3z - b(b - 1)yz + (1 - b)x^2yz + bxz^2 + x^3z^2), \end{aligned}$$

$$F_2 = \frac{(b - 1)(rx - y) + (b + 1)xz - 2x^2y}{(b + 1)((r - z)x - y)F_3(x)},$$

$$F_3 = \sqrt{\frac{4(b + x^2)}{(b + 1)^2} - 1}.$$

Proof. It is clear that the functions H_1 and H_2 are linearly independent, and that H_1 is a first integral of the Lorenz system. Now, a tedious computation (easy to do with the help of an algebraic manipulator such as maple or mathematica) shows that if X is the Lorenz vector field with $s = 0$, then H_2 satisfies $XH_2 = 0$, consequently H_2 is a first integral of the Lorenz system with $s = 0$. Hence, the proof of the proposition is complete. □

Now we will study the case $s \neq 0$. Since s is a parameter of the system, we can think of system (1) as the following system in the four variables x, y, z, s:

$$\dot{x} = s(y - x), \qquad \dot{y} = rx - y - xz, \qquad \dot{z} = -bz + xy, \qquad \dot{s} = 0. \tag{6}$$

A non-constant function $f = f(x, y, z, s)$ is a first integral of system (6) if

$$s(y - x)\frac{\partial f}{\partial x} + (rx - y - xz)\frac{\partial f}{\partial y} + (-bz + xy)\frac{\partial f}{\partial z} = 0. \tag{7}$$

Note that a function $f = f(s)$ different from a constant is a first integral of system (6), but it is not a first integral of the Lorenz system (1).

A *formal first integral* of the Lorenz system (1) is a non-constant formal power series f which satisfies that $Xf = 0$, where X is the vector field (2).

Theorem 4. *Suppose that $s \neq 0$ and b is not a negative rational. If $f = f(x, y, z, s)$ is a formal power series first integral of system (6), then f is a formal power series in the variable s.*

Proof. We assume that $f = f(x, y, z, s)$ is a formal power series first integral of system (6). We can think f as a power series in the variable s with coefficients power series in the variables x, y and z. Then, $f(x, y, z, 0)$ is a formal power series first integral of the Lorenz system (1) with $s = 0$. Since now we are in the assumptions of proposition 2, we can apply it and get that really $f(x, y, z, 0) = h(x)$, i.e., $f(x, y, z, 0)$ is a formal power series which is only a function of x. Therefore, since $f = f(x, y, z, s)$ is a formal power series in its variables, we always can write $f = h + sg$, where $h = h(x)$ and $g = g(x, y, z, s)$ is a formal power series in its variables. Then, since f is a first integral, it satisfies (7). So, after dividing by s the functions h and g satisfy the equation

$$(y-x)\left[\frac{\mathrm{d}h}{\mathrm{d}x} + s\frac{\partial g}{\partial x}\right] + (rx - y - xz)\frac{\partial g}{\partial y} + (-bz + xy)\frac{\partial g}{\partial z} = 0. \tag{8}$$

Now, we write $\overline{g} = g(x, y, z, 0)$ and restrict equation (8) to $s = 0$. Then, we get

$$(y-x)\frac{\mathrm{d}h}{\mathrm{d}x} + (rx - y - xz)\frac{\partial \overline{g}}{\partial y} + (-bz + xy)\frac{\partial \overline{g}}{\partial z} = 0. \tag{9}$$

Evaluating (9) on the points of the curve

$$(x, y, z) = \left(x, \frac{brx}{b + x^2}, \frac{rx^2}{b + x^2}\right),$$

which satisfy $rx - y - xz = -bz + xy = 0$, we have that (9) has the form

$$\frac{x}{b + x^2}(br - b - x^2)\frac{\mathrm{d}h}{\mathrm{d}x} = 0,$$

which clearly implies $\mathrm{d}h/\mathrm{d}x = 0$; i.e. h is a constant and we write $h = h(0)$. Therefore, (9) becomes

$$(rx - y - xz)\frac{\partial \overline{g}}{\partial y} + (-bz + xy)\frac{\partial \overline{g}}{\partial z} = 0.$$

Hence, $\overline{g}$ is a formal power series first integral of system (1) with $s = 0$, and by assumptions additionally we have that b is not a negative rational. So, from proposition 2, we obtain that $\overline{g} = g(x, y, z, 0) = \overline{g}(x)$. Consequently, we have that $g = \overline{g}(x) + sR$, where $R = R(x, y, z, s)$ is a formal power series in its variables. Then, $f = h(0) + s\overline{g}(x) + s^2 R$. Using that f satisfies (7), we get the equation

$$(y-x)\left[\frac{\mathrm{d}\overline{g}}{\mathrm{d}x} + s\frac{\partial R}{\partial x}\right] + (rx - y - xz)\frac{\partial R}{\partial y} + (-bz + xy)\frac{\partial R}{\partial z} = 0$$

i.e., $\overline{g}$ and R satisfy (8) replacing h by $\overline{g}$ and g by R. The same arguments used for h and g imply now that $\overline{g} = g(0)$ and $R = \overline{R}(x) + sS(x, y, z, s)$. Repeating this procedure inductively, we get that $f = f(s)$, which ends the proof of the theorem. □

From theorem 4 we get immediately the following result for the Lorenz system:

Corollary 5. *Suppose that $s \neq 0$ and b is not a negative rational. Then, the Lorenz system (1) has no formal power series first integrals. In particular, it has no analytic first integrals in a neighbourhood of the origin.*

Of course, if $s \neq 0$ and b is not a negative rational, then a global analytic first integral of the Lorenz system defined in $\mathbb{R}^3$ cannot exist, because in particular it must exist in a neighbourhood of the origin and this is in contradiction with corollary 5.

Corollary 6. *Suppose that $s \neq 0$ and b satisfies the non-resonance condition*

$$k_1 b + k_2(1+s) \neq 0, \qquad \text{for all} \quad k_1, k_2 \in \mathbb{Z}^+ \quad \text{with} \quad k_1 + k_2 > 0.$$

Then, the Lorenz system (1) does not have any analytic first integrals in a neighbourhood of the origin.

Proof. Note that the origin is a singular point for the Lorenz system. If $s \neq 0$ and $b \neq 0$, then the corresponding eigenvalues are

$$\lambda_1 = -b, \qquad \lambda_{2,3} = -\tfrac{1}{2}\left(1 + s \pm \sqrt{(1-s)^2 + 4rs}\right),$$

and all are different from zero. Now, suppose that there exist k_1, k_2 and k_3 non-negative integers such that $k_1 + k_2 + k_3 > 0$ and $k_1\lambda_1 + k_2\lambda_2 + k_3\lambda_3 = 0$. If in this last equality we only want that the parameters b and s should appear as in the statement of theorem 4, then we must choose $k_2 = k_3$. Hence, that equality becomes $k_1 b + k_2(1+s) = 0$. So, by theorem 1, it follows the corollary. □

Either corollary 5 or corollary 6 is not included because both conclusions are actually very similar, but assumptions differ. Thus, for instance, if

$$s = -1 - \frac{k_1}{k_2} b \neq 0,$$

with $k_1, k_2 \in \mathbb{Z}^+$, $k_1 + k_2 > 0$ and b is different from a negative rational number, corollary 5 holds, but corollary 6 cannot be applied.

On the other hand, if $s \neq 0$, b is a negative rational and $k_1 b + k_2(1+s) \neq 0$, for all $k_1, k_2 \in \mathbb{Z}^+$ with $k_1 + k_2 > 0$, then corollary 6 holds, but corollary 5 cannot be applied.

Acknowledgments

The first author is partially supported by a MCYT grant BFM2002-04236-C02-02, and a CIRIT grant number 2001SGR 00173. The second author has been supported by the Portuguese grant FCT/SFRH/BDP/14404/2003, and the Center for Mathematical Analysis, Geometry, and Dynamical Systems, Lisbon.

References

[1] Cairó L and Hua D 1993 Comments on: 'integrals of motion for the Lorenz system' *J. Math. Phys.* **34** 4370–1

[2] Furta S D 1996 On non–integrability of general systems of differential equations *Z. Angew Math. Phys.* **47** 112–31

[3] Giacomini H J, Repetto C E and Zandron O P 1991 Integrals of motion for three-dimensional non-Hamiltonian dynamical systems *J. Phys. A: Math. Gen.* **24** 4567–74

[4] Goriely A 1996 Integrability, partial integrability, and nonintegrability for systems of ordinary differential equations *J. Math. Phys.* **37** 1871–93

[5] Gupta N 1993 Integrals of motion for the Lorenz system *J. Math. Phys.* **34** 801–4

[6] Kús M 1983 Integrals of motion for the Lorenz system *J. Phys. A: Math. Gen.* **16** 689–91

[7] Llibre J and Zhang X 2002 Invariant algebraic surfaces of the Lorenz systems *J. Math. Phys.* **43** 7613–35

[8] Li W, Llibre J and Zhang X 2003 Local first integrals of differential systems and diffeomorphisms *Z. Angew Math. Phys.* **54** 235–55

[9] Lorenz E N 1963 Deterministic nonperiodic flow *J. Atmos. Sci.* **20** 130–41

[10] Lorenz E N 1993 The essence of chaos *The Jessie and Jonn Danz Lectures* (Seattle, WA: University of Washington Press)

[11] Poincaré H 1891 Sur l'intégration des équations différentielles du premier order et du premier degré I and II *Rend. Circolo Mat. Palermo* **5** 161–91

Poincaré H 1897 *Rend. Circolo Mat. Palermo* **11** 193–239

[12] Schwarz F 1985 An algorithm for determining polynomial first integrals of autonomous systems of ordinary differential equations *J. Symbol. Comput.* **1** 229–33

[13] Segur H 1982 Soliton and the inverse scattering transform *Topics in Ocean Physics* ed A R Osborne and P Malanotte Rizzoli (Amsterdam: North–Holland) pp 235–77

[14] Sparrow C 1982 *The Lorenz Equations* (New York: Springer)

[15] Steeb W H 1982 Continuous symmetries of the Lorenz model and the Rikitake two-disc dynamo system *J. Phys. A: Math. Gen.* **15** 389–90

[16] Strelcyn J M and Wojciechowski S 1988 A method of finding integrals for three-dimensional systems *Phys. Lett.* A **133** 207–12

[17] Swinnerton-Dyer P 2002 The invariant algebraic surfaces of the Lorenz system *Math. Proc. Camb. Phil. Soc.* **132** 385–93

[18] Zhang X 2002 Exponential factors and Darbouxian first integrals of the Lorenz system *J. Math. Phys.* **43** 4987–5001